LINEAR PROGRAMMING PROBLEM

ASHA RANI

LINEAR PROGRAMMING

- **FROM MATHEMATICAL VIEW :-** It is an optimization technique having one objective that is to be either maximized or minimized subject to a number of constraints and is having the following structure :

MAX. OR MIN. $Z = C_1X_1 + C_2X_2 + \ldots + C_nX_n$

SUBJECT TO

$a_1 X_1 + a_2 X_2 + \ldots a_n X_n \leq OR \geq OR = bi$

$X_1, X_2, \ldots, X_n \geq 0$

LINEAR PROGRAMMING

- **FROM ECONOMICS POINT OF VIEW :-** Linear programming is an optimization technique that helps in ensuring optimum allocation of scarce economic resources among various economic activities.

- It has only one objective to be either minimized or maximized subject to a number of economic constraints.

- It involves only linear function.

Assumptions of Linear Programming..

IT IS A SINGLE OBJECTIVE OPTIMIZATION TECHNIQUE - THAT IS, ONE OBJECTIVE IS TO BE EITHER MAXIMISED OR MINIMIZED.

THE OBJECTIVE IS TO BE OPTIMIZED SUBJECT TO A NUMBER OF CONSTRAINTS.

THE NUMBER OF VARIABLES AND CONSTRAINTS MUST BE FINITE.

THE RELATIONS/FUNCTIONS USED IN IT ARE LINEAR.

Assumptions of Linear Programming (continued…)

THE DECISION VARIABLES MUST BE NON-NEGATIVE REAL NUMBERS.

THE CONSTRAINTS MUST BE EITHER $\leq$, $\geq$ OR = TYPE.

THE OBJECTIVE FOLLOWS ADDITIVE PROPERTY.

THERE IS COMPLETE CERTAINTY.

Main Assumptions are…

Certainty exist.

Proportionality in the objective function and constraints (1 unit – 2 Kgs of Raw Material, 5 units 10 Kgs.).

Additivity (the total of all activities equals the sum of the individual activities).

Divisibility assumption that solutions need not necessarily be in whole numbers (integers).

Knowing history of Linear Programming

George B. Dantzig

- ▶ **LPP arose as a mathematical model developed during World War II to plan expenditures and returns in order to reduce costs to the army and increase losses to the enemy.**

- ▶ **It was kept secret until 1947. Postwar, many industries found its use in their daily planning.**

- ▶ **Many practical problems in operations research can be expressed as linear programming problems.**

- ▶ **Certain special cases of linear programming, such as network flow problems and multi-commodity flow problems are considered important enough to have generated much research on specialized algorithms for their solution.**

Steps involved in Linear Programming...

PROBLEM

FORMULATE IT AS LP MODEL

SOLVE THE MODEL

TEST THE SOLUTION

ACCEPT THE DECISIONS FOR ACTION
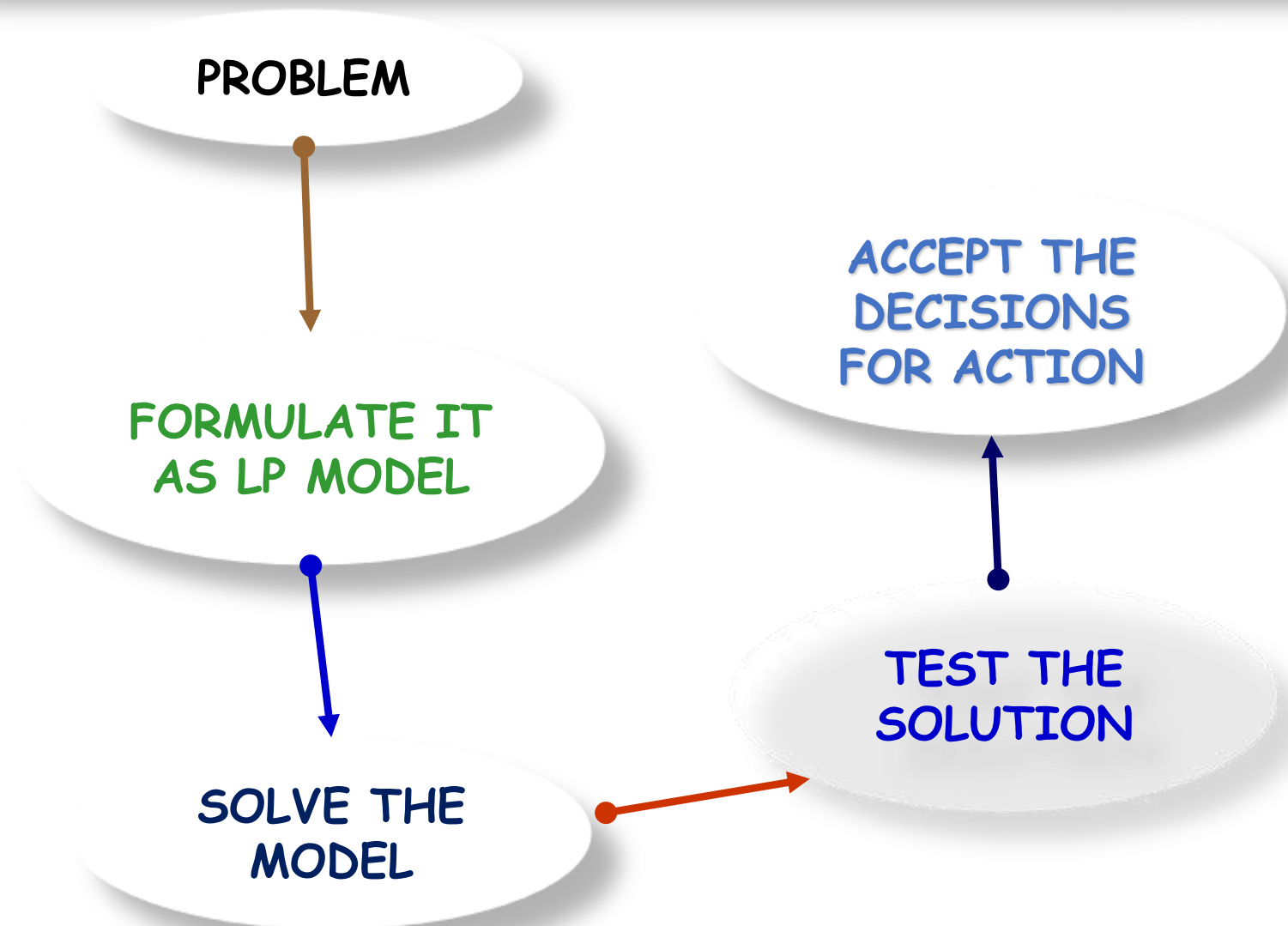
FORMULATION...

Identify the objective.

Identify the Problem that is to be decided.

Identify the variables about which a decision is to be taken.

Define the objective function.

Identify constraints and formulate them one by one.

Finally, introduce non-negative constraint.

FORMULATION... Problem No.1

1.   A producer can produce three products 701, 702 and 703 which require different manufacturing processes   - Grinding, Turning, Assembly and Testing. The manufacturing requirements of these products are given below:

| Processes	Products		
	701	**702**	**703 (per unit)**
Grinding	1hr.	2hrs.	3hrs.
Turning	3	1	2
Assembly	6	3	2
Testing	5	4	4

The available capacity of these operations are 40, 50, 60 and 80 hours respectively. The products contribute Rs. 25/-, Rs. 40/- and Rs. 100/- per unit respectively. Make a suitable mathematical model.

HOW TO SOLVE A LINEAR PROGRAMMING MODEL???

There are two widely used methods of solving a Linear Programming model -

- One is **GRAPHICAL METHOD**; and
- another is **SIMPLEX METHOD**.

Graphical Method…
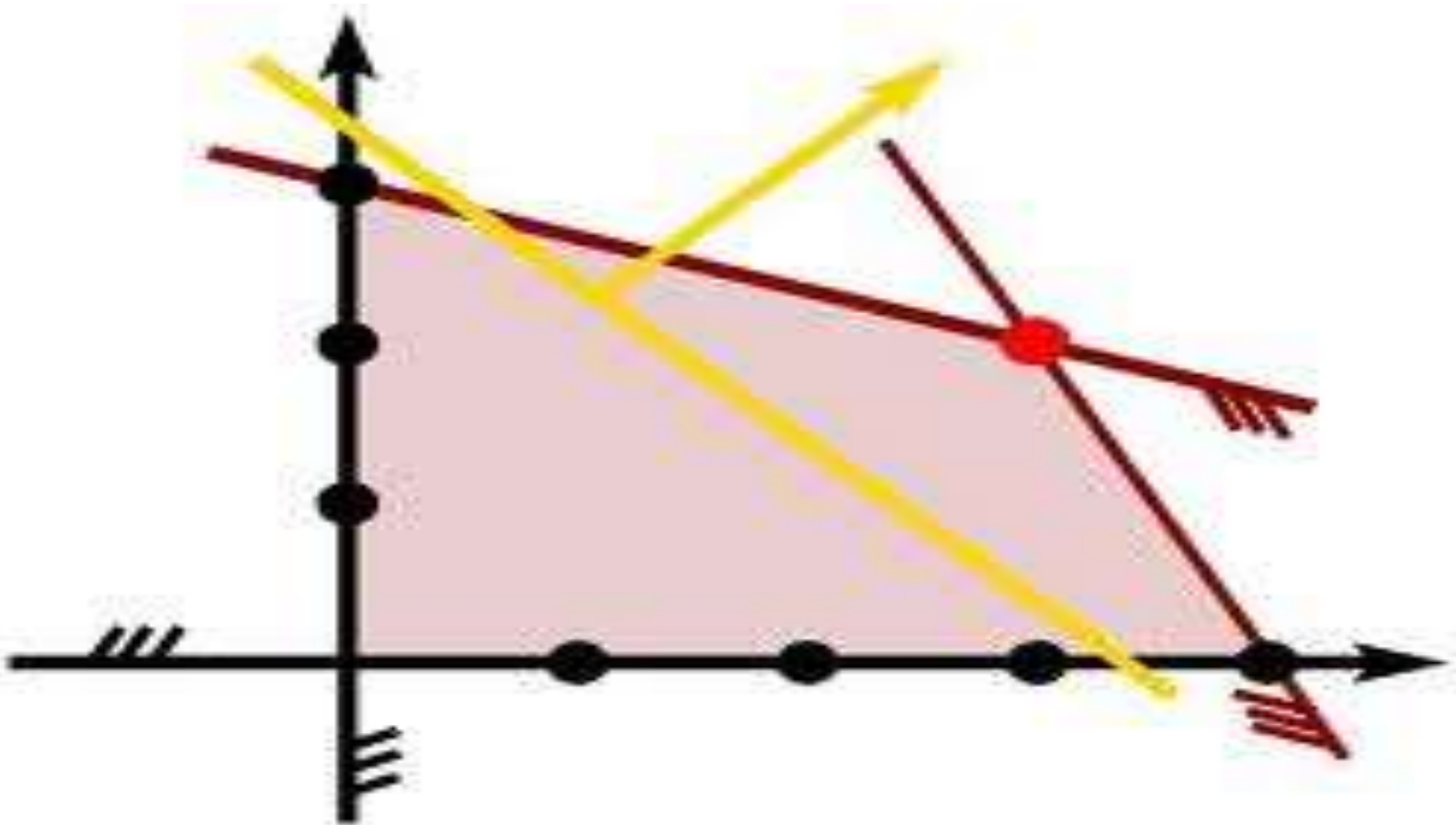
IMPORTANT THEOREMS …

While obtaining the optimum feasible solution to the linear programming problem, the statement of the following four important theorems are used:-

Theorems I: The feasible solution space constitutes a convex set.

Theorems II: Within the feasible solution space, basic feasible solution correspond to the extreme (or Corner) points of the feasible solution space.

IMPORTANT THEOREMS …

Theorem III: There are a finite number of basic feasible solution within the feasible solution space.

Theorem IV: The optimum feasible solution, if it exists. will occur at one, or more, of the extreme points that are basic feasible solutions. (Extreme Point Theorem)

Note. Convex set is a polygon "Convex" implies that if any two points of the polygon are selected arbitrarily then straight line segment Joining these two points lies completely within the polygon. The extreme points of the convex set are the basic solution to the linear programming problem.

Remember <u>Extreme Point Theorem</u> while solving a Linear Programming using Graphical Method

If an optimal solution of a Linear Programming problem exists, then it exists <u>only on corners of a feasible region</u>. In other words, an optimal solution to an LP problem can be found at an extreme point of the feasible region.
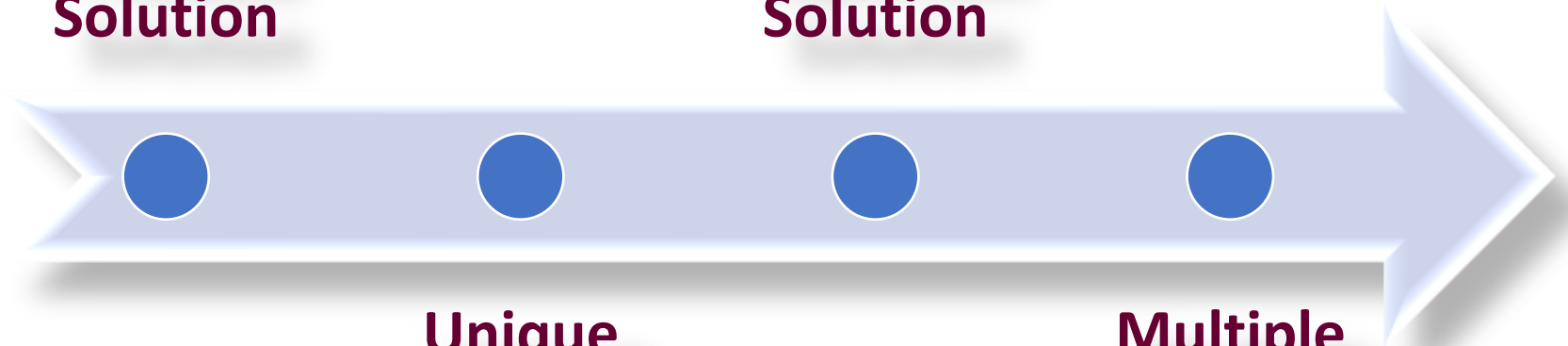
Identifying Special Cases... Graphical Method ...(About Solutions)
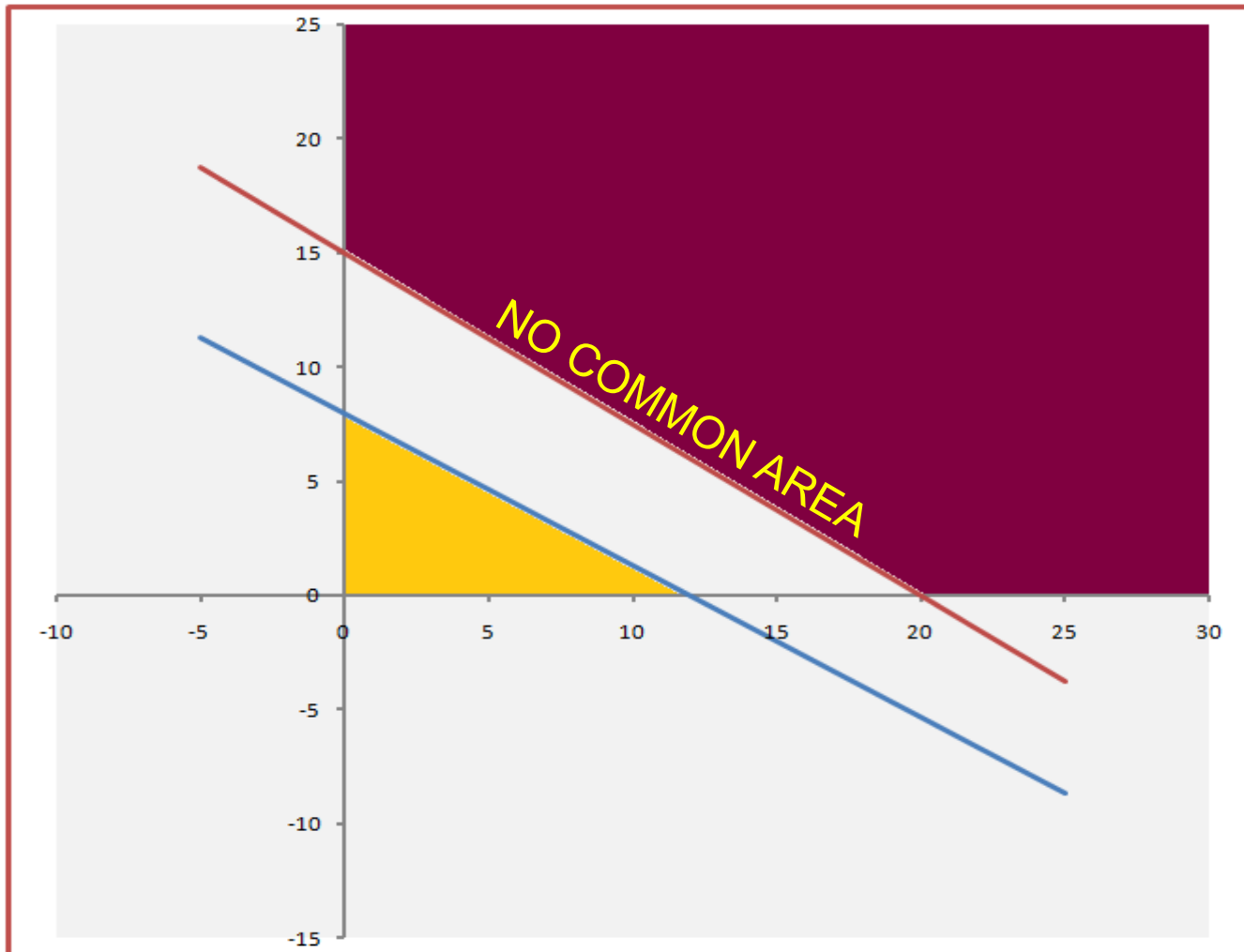
No Feasible Solution

No finite Optimum Solution

Unique Solution

Multiple Solutions

Infeasible Solution
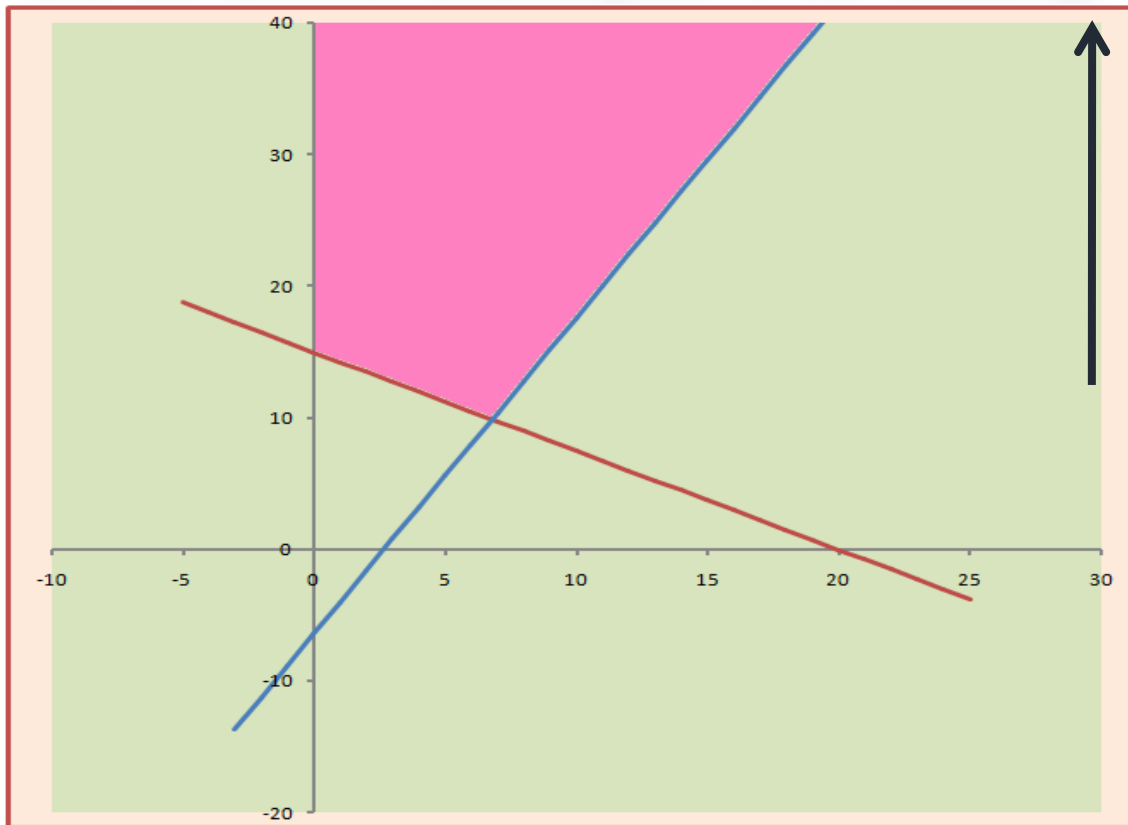
No Finite Optimum Solution

- Assume that a problem has an Objective Function: Max. Z = 20 X + 30 Y and the common area as given in the following:

In this case, we can make our Y as large as possible and then we can find that Z is tending towards infinity.

Multiple Optimum Solution ...

OPTIMUM SOLUTION Z = 120

12, 6

20, 0
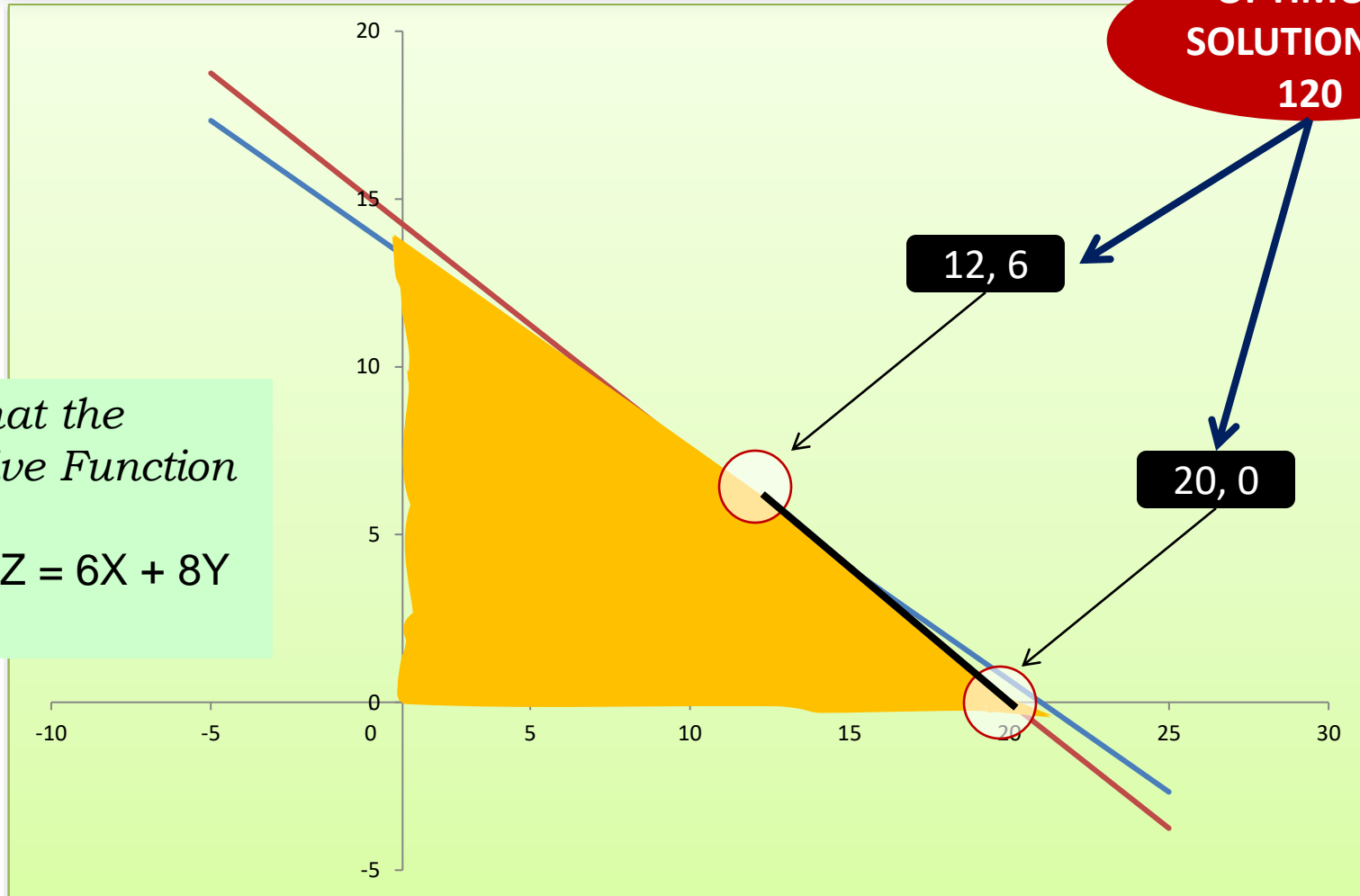
Note that the Objective Function is –

MAX. Z = 6X + 8Y

Now, let's move to…

Algebraic Method of Solving a LP Problem…

Simplex Method

- It is based on <u>SYSTEM OF EQUATIONS.</u> It is an optimization technique which searches for an optimum solution using System of Linear Equations.

> *Therefore, using simplex method requires conversion of all inequalities into equalities!*

How to convert '$\leq$' to '='?

Introduce Slack Variables.

It is a Non-Negative Variable which is to be added on the LHS of a '$\leq$' constraint to make it '=' to constraint.

For example: A constraint $3X_1 + 5X_2 + 10X_3 \leq 100$ can be converted into an '=' constraint by adding a SLACK VARIABLE (**S**); that is $3X_1 + 5X_2 + 10X_3 + S = 100$

How to convert '>' to '='?

Introduce <u>Surplus</u> Variables.

It is a Non-Negative Variable which is to be deducted from the LHS of a '>' constraint to make it '=' to constraint.

For example: A constraint $2X_1 + 6X_2 + 5X_3 \geq 60$ can be converted into an '=' constraint by adding a SURPLUS VARIABLE (**s**); that is $2X_1 + 6X_2 + 5X_3 + \mathbf{s} = 60$

Once, all constraints are reduced to 'equal to', present the problem in …

STANDARD FORM – A form in which all the constraints and functions are presented as 'equal to'; every variable is present in all equations; all variables are present in the same sequence and all variables are non-negative.

```
┌─────────────────────────┐
│   Start with a Basic    │
│    Feasible Solution    │
└─────────────────────────┘
              │
              ▼
          ╱╲
        ╱     ╲
      ╱  Is this ╲
    ╱ Basic Feasible ╲────── Yes ──────┐
    ╲   Solution    ╱              ┌─────────┐
      ╲ Optimal ? ╱               │  Stop   │
        ╲     ╱                   └─────────┘
          ╲╱
           │
          No
           │
           ▼
┌─────────────────────────┐
│  Find another Basic     │
│   Feasible Solution     │
└─────────────────────────┘
```

Iterate
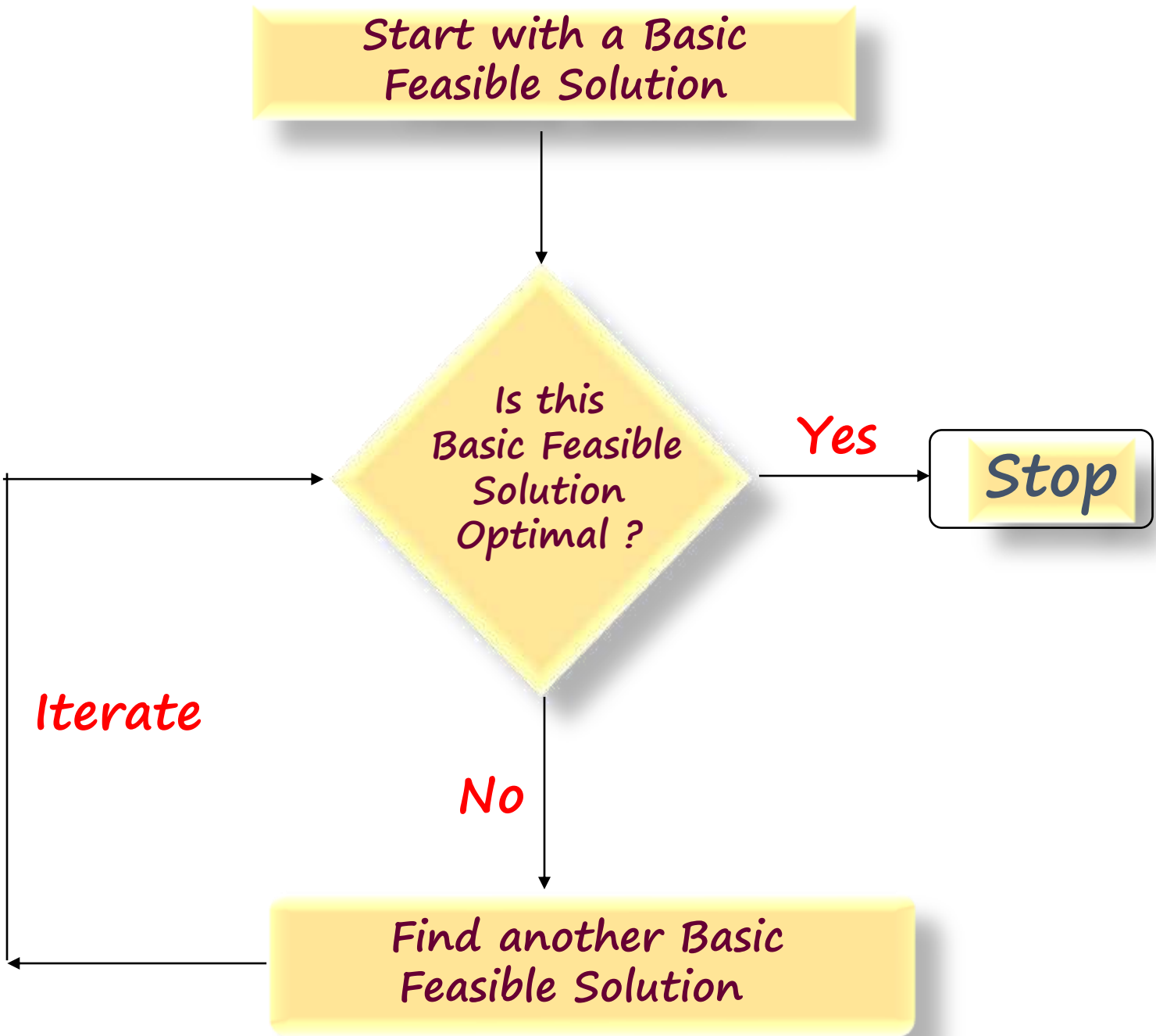
Understand ...

What is a FEASIBLE SOLUTION?

What is a BASIC FEASIBLE SOLUTION?

What is a BASIC VARIABLE?

What is a NON-BASIC VARIABLE?

FEASIBLE SOLUTION...

A solution is said to be feasible if it satisfies all the constraints including the non-negative constraints.

BASIC FEASIBLE SOLUTION...

Whenever the number of variables(n) are more than the number of constraints(m), there exist infinite solutions. If (n-m) number of variables have been put equal to zero, then the solution obtain through the remaining non-zero variables is called **BASIC FEASIBLE SOLUTION**.

NON-BASIC VARIABLE...

Variables that have been put equal to zero to get Basic Feasible Solution are called **Non-Basic Variables**.

Note that the non-basic variables are always zero by design.

BASIC VARIABLE

Variables that have **NOT** been put equal to zero to get Basic Feasible Solution are called **Basic Variables**.
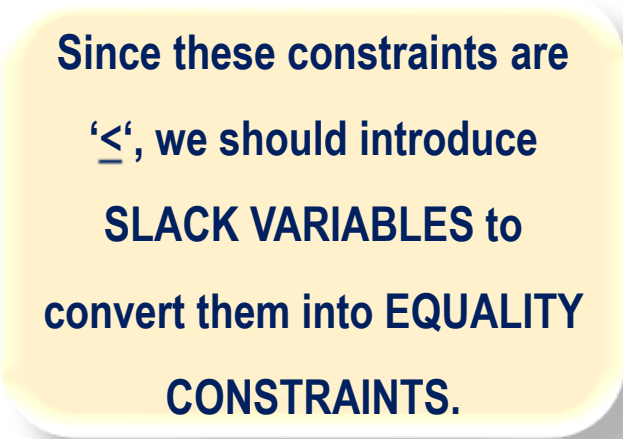
Example…

Max. $Z = 100X_1 + 60X_2 + 40X_3$

Subject to

$$X_1 + X_2 + X_3 \leq 100$$
$$10X_1 + 4X_2 + 5X_3 \leq 600$$
$$2X_1 + 2X_2 + 6X_3 \leq 100$$

$$X_1, X_2, X_3 \geq 0$$

Since these constraints are '$\leq$', we should introduce **SLACK VARIABLES** to convert them into **EQUALITY CONSTRAINTS.**

Converted Constraints...

$$X_1 + X_2 + X_3 + S_1 = 100$$

$$10X_1 + 4X_2 + 5X_3 + S_2 = 600$$

$$2X_1 + 2X_2 + 6X_3 + S_3 = 100$$

Standard Form...

Max. $Z = 100X_1 + 60X_2 + 40X_3 + 0S_1 + 0S_2 + 0S_3$

Subject to

$$X_1 + X_2 + X_3 + S_1 + 0S_2 + 0S_3 = 100$$

$$10X_1 + 4X_2 + 5X_3 + 0S_1 + S_2 + 0S_3 = 600$$

$$2X_1 + 2X_2 + 6X_3 + 0S_1 + 0S_2 + S_3 = 100$$

$$X_1, X_2, X_3, S_1, S_2, S_3, \geq 0$$

Make a Simplex Table...

Cj --->										
Cj	BASIS	bi	X1	X2	X3	S1	S2	S3	S4	S5
	X1									
	X2									
	S3									
	S4									
	S5									
	Zj									
		Cj-Zj								

Also, note that Cj – Zj row is also called Net Evaluation Row (NER).

Before we start solving a problem with Simplex Method, we have to make sure that ...

We get an **IDENTITY MATRIX** initially to obtain **BASIC FEASIBLE SOLUTION!**

Optimality Test of Maximization Problem...

All C_j-Z_j elements should be either zero or negative; that is no element in C_j-Z_j row should be positive.

Optimality Test of Minimization Problem

All Cj-Zj elements should be either zero or positive; that is no element in Cj-Zj row should be negative.

"BIG M" Method

This method is used whenever we are using Artificial Variables!

Now we will discuss PRIMAL and DUAL in LPP

Primal and Dual ...

A given problem of Linear Programming is known as PRIMAL while its mirror image is called DUAL.

PRIMAL	DUAL
Maximization	Minimization
No. of Decision Variables	No. of Constraints
No. of Constraints	No. of Dual Variables
C_js	b_is
Matrix A = $[a_{ij}]$	Transpose of A
$\leq$	$\geq$
$\geq$	$\leq$

Let's consider the following Linear Programming Problem

Maximize $Z = 32X_1 + 35X_2 + 45X_3$ (profit, Rs.)

subject to

$2X_1 + 3X_2 + 2X_3 \leq 120$ (operation 1, hr)

$4X_1 + 3X_2 + X_3 \leq 160$ (operation 2, hr)

$3X_1 + 2X_2 + 4X_3 \leq 100$ (operation 3, hr)

$X_1 + X_2 + X_3 \leq 40$ (storage, Ft^2)

$X_1 , X_2 , X_3 \geq 0$

Solving the Problem we get the following final Simplex Table

			32	35	45	0	0	0	0
Cj	**BASIC VARIABLES**	**bi**	**X1**	**X2**	**X3**	**S1**	**S2**	**S3**	**S4**
0	S1	10	-0.5	0	0	1	0	0.5	-4
0	S2	60	2	0	0	0	1	1.0	-5
45	X3	10	0.5	0	1	0	0	0.5	-1
35	X2	30	0.5	1	0	0	0	-0.5	2
	Zj	1500	40	35	45	0	0	5	25
		Cj - Zj	-8	0	0	0	0	-5	-25

Final - Simplex Table - Primal Problem

Let's consider the DUAL of Linear Programming Problem

Minimize $C^* = 120\,C_1 + 160\,C_2 + 100\,C_3 + 40\,C_4$

Subject to

$\quad 2\,C_1 + 4\,C_2 + 3\,C_3 + C_4 \geq 32 \qquad$ [Constraint related to Product#1]

$\quad 3\,C_1 + 3\,C_2 + 2\,C_3 + C_4 \geq 35 \qquad$ [Constraint related to Product#2]

$\quad 2\,C_1 + \quad C_2 + 4\,C_3 + C_4 \geq 45 \qquad$ [Constraint related to Product#3]

$\qquad\qquad C_1, C_2, C_3, C_4 \geq 0$

Solving the Problem we get the following final Simplex Table

			120	160	100	40	0	0	0
Cj	BASIC VARIABLES	bi	C1	C2	C3	C4	s1	s2	s3
100	C3	5	-0.5	-1	1	0	0	0.5	-0.5
40	C4	25	4	5	0	1	0	-2.0	1
0	s1	8	0.5	-2	0	0	1	-0.5	-0.5
	Zj	1500	110	100	100	40	0	-30	-10
		Cj - Zj	10	60	0	0	0	30	10

Final - Simplex Table - Dual Problem

Putting Primal and Dual Solution together…

Final - Simplex Table - Primal Problem

Cj	BASIC VARIABLES		32	35	45	0	0	0	0
		bi	X1	X2	X3	S1	S2	S3	S4
0	S1	10	-0.5	0	0	1	0	0.5	-4
0	S2	60	2	0	0	0	1	1.0	-5
45	X3	10	0.5	0	1	0	0	0.5	-1
35	X2	30	0.5	1	0	0	0	-0.5	2
	Zj	1500	40	35	45	0	0	5	25
	Cj - Zj		-8	0	0	0	0	-5	-25

Final - Simplex Table - Dual Problem

Cj	BASIC VARIABLES		120	160	100	40	0	0	0
		bi	C1	C2	C3	C4	s1	s2	s3
100	C3	5	-0.5	-1	1	0	0	0.5	-0.5
40	C4	25	4	5	0	1	0	-2.0	1
0	s1	8	0.5	-2	0	0	1	-0.5	-0.5
	Zj	1500	110	100	100	40	0	-30	-10
	Cj - Zj		10	60	0	0	0	30	10
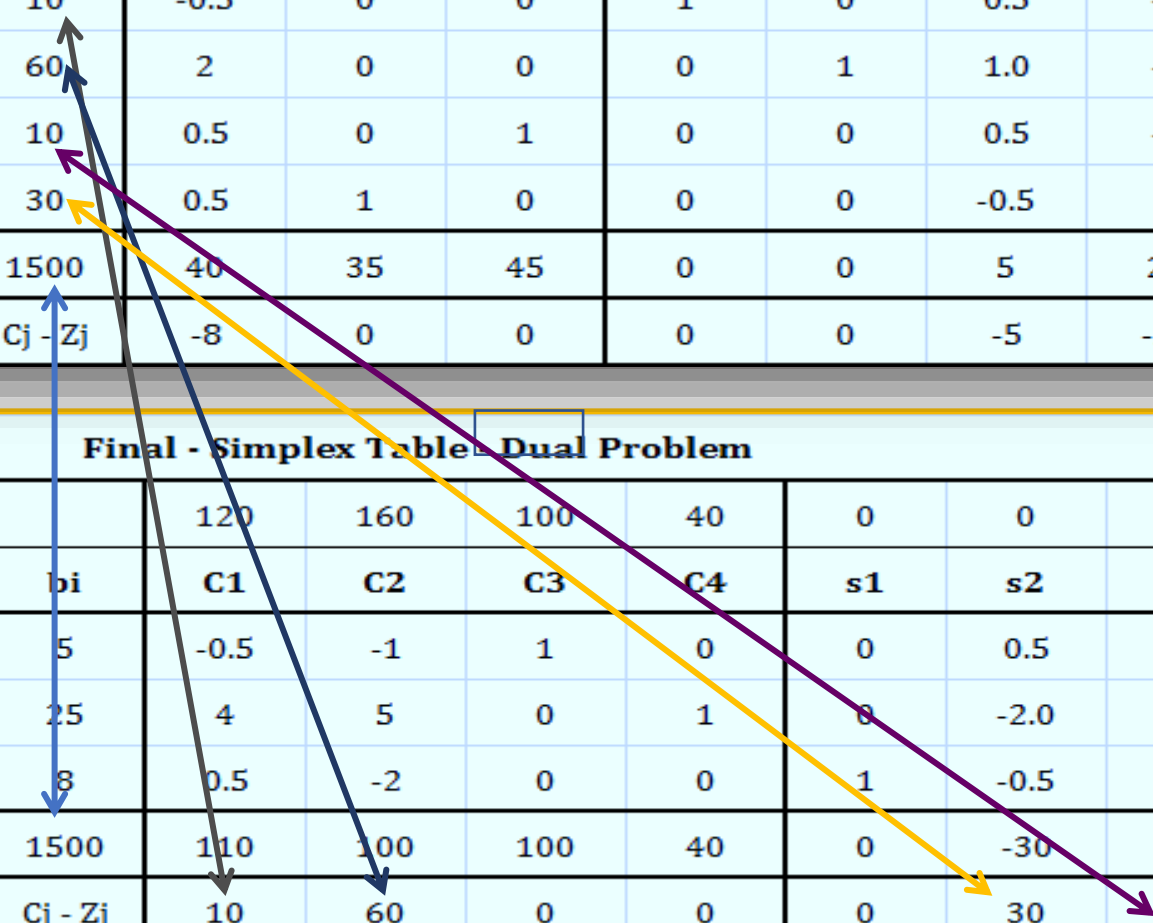
Relation between the solution of Primal and Dual

If Primal has an optimum solution, then dual will also have optimum solution; and the Optimum Value will be same.

If the Primal has unbounded solution, then dual will have infeasible solution.

If the Primal has infeasible solution, then dual will have unbounded solution.

Interpretations...

Dual Variables represent the OPPORTUNITY COST or the SHADOW PRICES of the resources.

C_j Coefficients represent the Contribution of variables towards the objective function.

$C_j - Z_j$ represents the marginal contribution of the variables.

Slack variables represent the UNUSED Resources.

Surplus variables represent OVER or EXCESS capacity or utilization.

Identifying Special Cases...Simplex Method

Infeasible Solution

Unbounded Solution

Degeneracy Problem

Multiple Optimal Solutions

Infeasible solution: This occur when artificial variable is present in the final simplex table.

Unbounded solution: A simplex problem said to have a unbounded solution when all the value of key column is either 0 or negative.

Degeneracy problem: This occur when there is tie in the departing variable. We may select any one of these quotient to revise the simplex table as a consequence to this in the next table the variable which is not selected by us is having 0 value in the solution value column corresponding to basic variable. In that situation we can say the basic feasible solution will degenerate.

Multiple Optimal Solution: This occur when non basic variable have 0 value in $C_J - Z_J$ row which indicates that the problem has more than one solution.

A farmer has 1000 acres of land on which he can grow corn, wheat or soya beans. Each acre of corn costs Rs. 100 for preparation, requires 7 man-days of work and yields a profit of Rs. 30. An acre of wheat costs Rs. 120 to prepare, requires 10 man-days of work and yields a profit of Rs. 40. An acre of soya beans costs Rs. 70 to prepare, requires 8 man-days of work and yields a profit of Rs. 20. If the farmer has Rs. 1,00,000 for preparation and can count on 8000 man-days of work, how many acres should be allocated to each crop to maximize profits? Formulate this as a LPP problem and solve it by using simplex method.

Practice Question:2

One of the methods used by *xyz* company to separate copper, lead and zinc from ores in the flotation separation process. This process consists of three steps: oiling, mixing and separation. These steps must be applied for 2, 2 and 1 hour respectively to produce 1 unit of lead, and 1, 1 and 3 hours respectively to produce 1 unit of zinc. The oiling and separation phases of the process can be in operation for a maximum of 10 hours a day, while the mixing phase can be in operation for a maximum of 11 hours a day. The *xyz* company makes a profit of Rs. 45 per unit of copper, Rs. 30 per unit lead and Rs. 35 per unit zinc. The demand for these metals is unlimited. How many units of each metal should be produced daily by use of the floating process to achieve the highest profit?

A diet for a sick person must contain atleast 4,000 units of vitamins, 50 units of minerals and 14 units of calories. Two foods *A* and *B* are available at a cost of Rs. 4 and Rs. 3 per unit respectively. If one unit of *A* contains 200 units of vitamins, 1 unit of mineral and 40 calories and one unit of food *B* contains 100 units of vitamins, 2 units of minerals and 40 calories, find by simplex method, what combination of food be used to have least cost.

Practice Question:4

A firm uses three machines in the manufacture of three products. Each unit of product *A* requires 3 hours on machine *I*, 2 hours on machine II and 1 hour on machine *III*. Each unit of product *B* requires 4 hours on machine *I*, 1 hour on machine II and 3 hours on machine III, while each unit of product *C* requires 2 hours on each of the three machines. The contribution margin of the three products is Rs. 30, Rs. 40 and Rs. 35 per unit respectively. The machine hours available on the three machines are 90, 54 and 93 respectively.

(i) Formulate the above as a linear programming problem and solve for maximum profit, using the simplex method.
(ii) Write the dual to the LPP.
(iii) Obtain the optimum values of the dual variables, and verify that the primal and the dual problems have the same objective function values.